Bjorn [illegible] S4114213

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1 a)

Let S_n be a statement about the positive integer n.

Suppose that:

5 1. S_1 holds
2. S_{k+1} holds whenever S_k holds.

Thus, using mathematical induction, we have proven that S_n holds for all positive integers n.

1b

$S_n: 1+5+9+\dots+(4n-3) = 2n^2-2$

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Given: $n \in \mathbb{N}$; $n \geq 1$

Proof by mathematical induction:

1 Base step: S_1: $4\cdot 1 - 3 = 2\cdot 1^2 - 1$

$4-3 = 2-1$

2 $1 = 1$

$\Downarrow$

S_1 holds

2 Induction step: 2 Assume: S_n holds

2 Must prove: S_{n+1} holds whenever S_n holds

$S_n: 1+5+9+\dots+(4n-3) = \underline{2n^2 - \cancel{2}n}$ (Assumed true)

$S_{n+1}: \underline{1+5+9+\dots+(4n-3)} + [4(n+1)-3] = 2(n+1)^2 - \cancel{2}(n+1)$

$\Downarrow$

$S_{n+1}: 2n^2 - n + [4(n+1)-3] = 2(n+1)^2 - \cancel{2}(n+1)$

7 $2n^2 - n + [4n+4-3] = 2(n^2+2n+1) - \cancel{2}(n+1)$

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$2n^2 - n + [4n+4-3] = 2(n^2+2n+1) - \cancel{2}(n+1)$

2 $2n^2 - n + 4n + 1 = 2n^2 + 4n + 2 - (\cancel{2}n+1)$

$2n^2 + 3n + 1 = 2n^2 + 4n + 2 - n - 1$

2 $\boxed{2n^2 + 3n + 1 = 2n^2 + 3n + 1} \rightarrow$ S_{n+1} holds whenever S_n holds

2 Thus, by using (the principle of) mathematical induction, we have proven that S_n holds for all natural numbers $n \geq 1$

Exercise 2 18

S_n: $13^n - 6^n$ divisible by 7

Given: $n \in \mathbb{Z}$; $n \geq 1$

Proof by mathematical induction

1 Base step: S_1: $13^1 - 6^1 = 13 - 6 = 7$ 2

$\frac{7}{7} = 1$ (therefore S_1 holds)

2 Induction step: Assume: S_n holds 2

Must prove: S_{n+1} holds whenever S_n holds 2

S_n: $13^n - 6^n$ is divisible by 7

S_{n+1}: $13^{n+1} - 6^{n+1}$ is divisible by 7

$13^{n+1} - 6^{n+1} = 13^n \cdot 13 - 6^n \cdot 6$ 1

$\cancel{= 13^n \cdot [2 \cdot 7 - 1] - 6^n \cdot (7-1)}$

$= 13^n \cdot (14 - 1) - 6^n \cdot (7-1)$

$= 13^n \cdot 14 - 13^n - 6^n \cdot 7 + 6^n$ 8

$= 13^n \cdot 14 - 6^n \cdot 7 - 13^n + 6^n$

$= 7(13^n \cdot 2 - 6^n) - (13^n - 6^n)$

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$= 7(13^n \cdot 2 - 6^n) - (13^n - 6^n)$ Assume S_n holds, therefor $13^n - 6^n$ is devisible by 7

$= 7\left(13^n \cdot 2 - 6^n - \frac{13^n - 6^n}{7}\right)$

$\Downarrow$ devisible by 7 $\rightarrow S_{n+1}$ holds

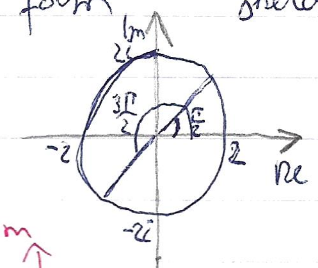
We have proven that S_{n+1} ~~is true~~ holds whenever S_n ~~is true~~ holds

Thus, using the principle of mathematical induction, we have proven that S_n holds for all positive integers $n \geq 1$ 3

Exercise 3 12

Solve $z^3 = -2 - 2i$

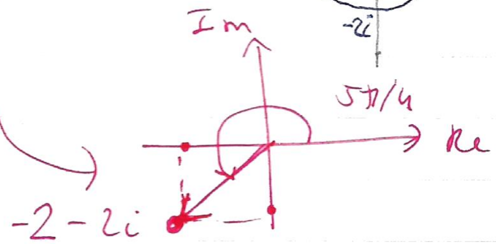
First we express $(-2-2i)$ in polar form 1 sketch of $(-2-2i)$



$r = \sqrt{(-2)^2 + (-2)^2} = \sqrt{8}$ 2

$\theta = \tan^{-1} \frac{-2}{-2} = \tan^{-1} 1$

$\theta = \frac{\pi}{2} + k \cdot \pi$



$\hookrightarrow$ so $\theta = 5\pi/4 + 2\pi k$

$\downarrow$

$(-2-2i) = \sqrt{8} \cdot e^{i\left(\frac{\pi}{2} + k\pi\right)}$ 2

$z^3 = \sqrt{8} \cdot e^{i\left(\cancel{\frac{\pi}{2}} + k\pi\right)}$

$z = \sqrt{2} \cdot e^{i\left(\cancel{\frac{\pi}{6}} + \frac{k\pi}{3}\right)}$ 2 where $k = 0; 1; 2$

$z = \sqrt{2} \cdot e^{i\left(\cancel{\frac{\pi}{6}} + 0\right)} \vee z = \sqrt{2} \cdot e^{i\left(\cancel{\frac{\pi}{6}} + \frac{\pi}{3}\right)} \vee z = \sqrt{2} \cdot e^{i\left(\cancel{\frac{\pi}{6}} + \frac{2\pi}{3}\right)}$

$z_1 = \sqrt{2} \cdot e^{i\frac{\pi}{6}}$ 1 $\vee$ $z_2 = \sqrt{2} \cdot e^{i\frac{\pi}{2}}$ 1 $\vee$ $z_3 = \sqrt{2} \cdot e^{i\frac{5\pi}{6}}$ 1

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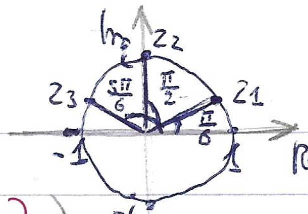
$r = \sqrt{2}$

3 - Sketch of solutions in complex plane



(-1)

2

Exercise 4

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$$e^{-iz} = \frac{1}{2i} \qquad \frac{1}{2i}\cdot\frac{i}{i} = \frac{i}{-2}$$

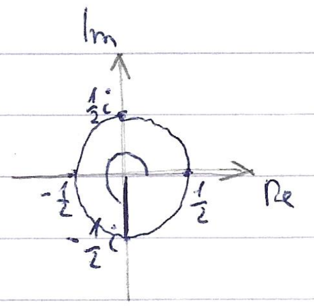
(standard form)

First we express $\left(\frac{i}{-2}\right)$ in polar form $\quad \cancel{\frac{1}{2}i = 0 + \frac{1}{2}i} \quad \frac{i}{-2} = 0 + \left(-\frac{1}{2}i\right)$

$$r = \sqrt{0^2 + \left(-\frac{1}{2}\right)^2}$$

$$\cancel{\theta = \frac{\pi}{2} + 2k\pi}$$

$$\boxed{r = \frac{1}{2} \qquad \theta = \frac{3\pi}{2} + 2k\pi}$$



$$\downarrow$$

$$\frac{i}{-2} = \frac{1}{2}e^{i\left(\frac{3\pi}{2} + 2k\pi\right)}$$

$$e^{-iz} = \frac{1}{2}e^{i\left(\frac{3\pi}{2} + 2k\pi\right)} \checkmark$$

(now we express z in standard form $z = a + bi$)

$$e^{-i(a+bi)} = \frac{1}{2}e^{i\left(\frac{3\pi}{2} + 2k\pi\right)}$$

$$e^{b}\cdot e^{-ia} = \frac{1}{2}e^{i\left(\frac{3\pi}{2} + 2k\pi\right)}$$

$$\downarrow$$

$$\boxed{e^{b} = \frac{1}{2} \qquad b = \ln\left(\frac{1}{2}\right)} \checkmark$$

$$e^{-ia} = e^{i\left(\frac{3\pi}{2} + 2k\pi\right)}$$

$$\downarrow$$

$$-ia = i\left(\frac{3\pi}{2} + 2k\pi\right)$$

$$\boxed{a = -\left(\frac{3\pi}{2} + 2k\pi\right)} \checkmark$$

$$\boxed{\begin{aligned} z &= a + bi \\ z &= -\left(\frac{3\pi}{2} + 2k\pi\right) + \ln\left(\frac{1}{2}\right)\cdot i \end{aligned}} \checkmark$$

for $k \in \mathbb{Z}$

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Exercise 5 15

$\lim_{x \to 1} x^2 = 1$ Prove using definition of limit.

(Def. of limit) $\forall \varepsilon > 0 \ \exists \delta > 0 : 0 < |x-a| < \delta \rightarrow |f(x) - L| < \varepsilon$ 5

Step 1 $\downarrow$ (Assume δ exists and it fulfills the conditions)

$0 < |x-1| < \delta \rightarrow |x^2 - 1| < \varepsilon$

$\underline{x - 1 < \delta}$ $|x^2 - 1^2| < \varepsilon$ 2

$|x-1||x+1| < \varepsilon \rightarrow |x^2-1| < \delta\,|x+1|$ (absolute value)

$\delta\,(\underline{x+1}) < \varepsilon$

$0 < x - 1 < \delta$ $\boxed{\delta = 1}$ (chosen number) 1

$0 < x - 1 < 1$

$1 < x < 2$ 1

$\delta\,|x+1| < \varepsilon$

$\delta\,|2+1| < \varepsilon$ 2

$\delta \cdot 3 < \varepsilon$

$\boxed{\delta = \frac{\varepsilon}{3}}$

$\delta = \min\{1; \frac{\varepsilon}{3}\}$ 2

Step 2: Substitute δ for $\min\{1; \frac{\varepsilon}{3}\}$

$\forall \varepsilon > \exists \delta > 0 : 0 < |x-a| < \delta \rightarrow |fx - L| < \varepsilon$ 1

$\delta = \min\{1; \frac{\varepsilon}{3}\}$

absolute value!!

$|x-1| < 1 \rightarrow |x^2 - 1^2| < \varepsilon$

$x - 1 = 1$ ✓ $|(x-1)(x+1)| < \varepsilon$

$(\delta = x-1 = 1)$ $|x-1| \cdot 3 < \varepsilon$

$\frac{\varepsilon}{\cancel{3}} \cdot \cancel{3} = \varepsilon$ ✓ $(\delta = x - 1 = \frac{\varepsilon}{3})$

$\boxed{\varepsilon = \varepsilon}$

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Therefor, by using the definition of a limit, we have proven that

$\lim_{x \to 1} x^2 = 1$ is true